Poynting Theorem and Poynting vector

The theorem states that "the rate at which electromagnetic energy in a finite volume decreases with time is equal to the rate of dissipation of energy in the form of Joule heat plus the rate at which energy flows out of the volume."

From the Maxwell's third and fourth electromagnetic equations,

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \text{—— ①}$$

&

$$\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t} \quad \text{—— ②}$$

Now

$$-\vec{H}\cdot(\nabla \times \vec{E}) = \vec{H}\cdot\frac{\partial \vec{B}}{\partial t} \quad \text{—— ③}$$

and

$$\vec{E}\cdot(\nabla \times \vec{H}) = \vec{E}\cdot\vec{J} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t} \quad \text{—— ④}$$

Adding ③ & ④ we get

$$-\vec{H}\cdot(\nabla\times\vec{E}) + \vec{E}\cdot(\nabla\times\vec{H}) = \vec{E}\cdot\vec{J} + \left[\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t}\right]$$

$$-\left[\vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H})\right] = \vec{E}\cdot\vec{J} + \left[\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t}\right]$$

$$-\nabla\cdot(\vec{E}\times\vec{H}) = \vec{E}\cdot\vec{J} + \left[\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} + \vec{E}\cdot\frac{\partial \vec{D}}{\partial t}\right] \quad \text{—— ⑤}$$

$$\left[\because \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B}) = \nabla\cdot(\vec{A}\times\vec{B})\right]$$

Now

$$\vec{H}\cdot\frac{\partial \vec{B}}{\partial t} = \mu\vec{H}\cdot\frac{\partial \vec{H}}{\partial t} = \frac{1}{2}\mu\frac{\partial}{\partial t}(\vec{H}\cdot\vec{H}) \quad [\because \vec{B} = \mu\vec{H}]$$

$$= \frac{1}{2}\frac{\partial}{\partial t}(\vec{H}\cdot\vec{B}) \quad \text{—— ⑥}$$

&

$$\vec{E}\cdot\frac{\partial \vec{D}}{\partial t} = \epsilon\vec{E}\cdot\frac{\partial \vec{E}}{\partial t} = \frac{1}{2}\epsilon\frac{\partial}{\partial t}(\vec{E}\cdot\vec{E}) \quad [\because \vec{D} = \epsilon\vec{E}]$$

$$= \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}\cdot\vec{D}) \quad \text{—— ⑦}$$

Now equation ⑤ becomes

$$\nabla\cdot(\vec{E}\times\vec{H}) = -\vec{E}\cdot\vec{J} - \frac{\partial}{\partial t}\frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})$$

Integrating over a volume V, then

$$\iiint_V \nabla\cdot(\vec{E}\times\vec{H})\,dv = -\iiint_V(\vec{E}\cdot\vec{J})\,dv - \frac{\partial}{\partial t}\iiint \frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})\,dv$$

If S be the surface enclosing volume V, then

$$-\frac{\partial}{\partial t}\iiint_V \frac{1}{2}(\vec{E}\cdot\vec{D}+\vec{H}\cdot\vec{B})\,dv = \iiint_V(\vec{J}\cdot\vec{E})\,dv + \oint_S(\vec{E}\times\vec{H})\,d\vec{s} \quad -⑧$$

(we have used Gauss divergence theorem)

Now the term

$$-\frac{\partial}{\partial t}\iiint_V \frac{1}{2}\left[(\vec{E}\cdot\vec{D})+(\vec{H}\cdot\vec{B})\right]dv$$

$$= -\frac{\partial}{\partial t}\left[\iiint_V \frac{1}{2}(\vec{E}\cdot\vec{D})\,dv + \iiint_V \frac{1}{2}(\vec{H}\cdot\vec{B})\,dv\right]$$

$$= -\frac{\partial}{\partial t}\left[U_e + U_m\right]$$

where

$$U_e = \iiint \frac{1}{2}(\vec{E}\cdot\vec{D})\,dv$$

&

$$U_m = \iiint_V \frac{1}{2}(\vec{H}\cdot\vec{B})\,dv$$

are the electric energy and magnetic energy stored in volume V

The left hand side of equation ⑧ represents the time rate of decrease of electromagnetic energy in volume V.

The term $\iiint_V(\vec{J}\cdot\vec{E})\,dv$ represents the rate of dissipation of energy in the form of Joule heat in V.

The term

$\oint (\vec{E} \times \vec{H}) \cdot d\vec{S}$ represents the time rate of ~~change~~ flow of energy from v through the surface S.

$$\vec{E} \times \vec{H} = \vec{S}$$ (do not Confuse with surface $\vec{S}$)

So $\vec{S} = \vec{E} \times \vec{H}$ is the energy flowing through unit area and unit time and is known as the Poynting vector.

Hence the Poynting vector ($\vec{S}$) may be defined as the amount of electromagnetic field energy flowing through unit area of the surface in a direction perpendicular to the plane containing $\vec{E}$ & $\vec{H}$ per unit time.